

Recursion Invariants

CSCI 432, Fall 2021

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Recursion Invariants: Proving Partial Correctness

Recall the HANOI algorithm (given in Algorithm 1); see [?, Ch. 1].

Algorithm 1 HANOI($n, src, dest, tmp$)

Input: $n \in \mathbb{N}$, and three towers with disks: $src, dest, tmp$ such that P

Output: R (see below)

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1: if  $n > 0$  then  
2:   HANOI( $n - 1, src, tmp, dest$ )  
3:   move top disk from  $src$  to  $dest$   
4:   HANOI( $n - 1, tmp, dest, src$ )  
5: end if
```

1. Suppose we have n disks total. What are the assumptions on the input to the initial call HANOI($n, src, dest, tmp$)?

Answer:

Going forward, we will call these assumptions P .

2. What does it mean for $\text{HANOI}(n, \text{src}, \text{dest}, \text{tmp})$ to execute correctly? What does it return / what does it need to accomplish?

Answer:

Going forward, we call this statement Q .

3. For a general call to the recursive algorithm what are the assumptions on the input (For convenience, suppose the call is: $\text{HANOI}(k, A, B, C)$).

Answer:

Note: when making a recursive call, we must justify that we have met these conditions.

4. What is the recursion invariant?

The recursion invariant is statement (i.e., a sentence that can be evaluated to TRUE/FALSE). In particular, R is (in general) the statement, *Each recursive call $\text{HANOI}(k, A, B, C)$ executes correctly.* What does that mean in our case? Well, it means

- There are currently no violations of smaller disks on larger disks. (Note: the world would crumble if this were violated at any time), AND
- the k smallest disks are now on B , AND
- no other disks besides the k smallest have moved (since right before this call).

We use the recursion invariant to prove INITIALIZATION, MAINTENANCE, and END. So, sometimes we may see the invariant right away. Other times, we may need to try the proofs then revisit the invariant (as we might realize that we forgot something).

5. **INITIALIZATION** This is like the base case of induction. Colloquially, we ask “Why is this true for the smallest input?” (And, what are those inputs that would allow us to return without a recursive call?) More formally, we can say: If $n_0 = 0$, then after the call to $\text{HANOI}(n_0, A, B, C)$, the recursion invariant R is satisfied.

Answer:

Note: sometimes, just as in induction, there may be more than one base case.

6. **MAINTENANCE:** This part is JUST like the inductive step of induction. Let $k \in \mathbb{N}$ such that $k \geq n_0$. Assume that, for all $k' \in \mathbb{N}$ such that $n_0 \leq k' \leq k$, the recursion invariant R holds after a call to $\text{HANOI}(k', *, *, *)$. (That was the equivalent to the inductive assumption). Now, we must prove that R holds after $\text{HANOI}(k + 1, A, B, C)$. (Hint: for this, we will almost always need to use the line numbers as we walk through the algorithm to explain how the maintenance step works).

Answer:

7. **END:** This is where we diverge from induction. Since algorithms are finite, we can't go on forever. Colloquially, we say “if the initial call $\text{HANOI}(n, src, dest, tmp)$ finishes executing, then all n disks (which were initially on src) have moved to dst .” More formally, we can phrase this as: If the recursion invariant holds, if the preconditions P are satisfied, and if the algorithm completes execution, then the post-condition Q is satisfied. (For shorthand, I might write $R \wedge P \wedge T \implies Q$).

Answer:

References